

On a Certain Combination Theorem

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1. Introduction and statement of result

In the study of kleinian groups it is important to construct new kleinian groups.

In this paper we shall show a simple combination theorem. Before stating our theorem, we explain some notation and definitions.

We denote the extended complex plane, or Riemann sphere, by $\hat{\mathbb{C}}$. If G is a group of Möbius transformations, and $z \in \hat{\mathbb{C}}$, then z is called a regular point for G ($z \in R(G)$) if there is a neighborhood U of z so that $g(U) \cap U = \emptyset$, for all $g \in G$, $g \neq 1$. If $R(G) \neq \emptyset$, then G is called a kleinian group. A set D is called fundamental set (FS) for G , if

- (a) $g(D) \cap D = \emptyset$, for all $g \in G$, $g \neq 1$,
- (b) there exist an element g in G and a point w in D such that $g(w) = z$ for any z in $R(G)$.

Our result is the following theorem.

THEOREM. *Let G be a kleinian group. Let $f \in G$ be an elliptic element with the period 2. Let D_1, D_2 be FS's for $G, \langle f \rangle$, respectively. Let r_1, r_2, r_3 be simple closed curves.*

Assume that

- (1) $r_2 \cap r_3$ is either empty or one point;
- (2) $r_1 \cap r_i$ ($i=2, 3$) is only two points;
- (3) $\text{Ext } r_1 \subsetneq D_2 \subsetneq (\text{Int } r_1)^c$;
- (4) there exists g in G such that $g(r_2) = r_3$;
- (5) $f(r_2) = r_3$;
- (6) $(\text{Int } r_1) \setminus [(\overline{\text{Int } r_1}) \cap (\text{Int } r_2)] \setminus [(\overline{\text{Int } r_1}) \cap (\text{Int } r_3)] \subsetneq D_1 \subsetneq (\text{Ext } r_2) \cap (\text{Ext } r_3)$;
- (7) $D_1 \cap D_2 \neq \emptyset$;

Then

- (I) Γ , the group generated by G and f , is kleinian;

(II) $D = D_1 \cap D_2$ is FS for Γ .

2. Proof of theorem

First we shall show that two points in $D = D_1 \cap D_2$ are not equivalent under Γ . Let z be a point in D . We divide into five cases.

i) Since $z \in D_1$, $g(z) \in D_1^c$ for any $g \in G \setminus \{1\}$. Hence $g(D) \cap D = \emptyset$ for any $g \in G \setminus \{1\}$.

ii) As $z \in D_2$, $f(z) \in D_2^c$. Therefore $f(D) \cap D = \emptyset$.

iii) We consider an element which is of the form hf , where $h \in G$. Since $f(z) \in D_2^c$, $f(z) \in \text{Int } r_1$. If $f(z) \in (\text{Int } r_1) \cap (\text{Int } r_2)$, then $z \in (\text{Int } r_3) \cap (\text{Ext } r_1)$. Hence $z \notin D_1 \cap D_2$. This contradicts our assumption that z is a point in D . Therefore $f(z) \notin (\text{Int } r_1) \cap (\text{Int } r_2)$. If $f(z) \in (\text{Int } r_1) \cap (\text{Ext } r_2) \cap (\text{Ext } r_3)$, then $hf(z) \in D_1^c$. Hence $hf(D) \cap D = \emptyset$. If $f(z) \in D$, then $z \in (\text{Int } r_2) \cap (\text{Ext } r_1)$. Hence $z \notin D_1 \cap D_2$. This is a contradiction. We conclude that $hf(D) \cap D = \emptyset$.

iv) In this case, we show that $fh(D) \cap D = \emptyset$ for every h in G . If $h(z) \in (\text{Int } r_2) \cup (\text{Int } r_3)$, then $fh(z) \in (\text{Int } r_2) \cup (\text{Int } r_3)$. Hence $fh(z) \in (D_1 \cap D_2)^c$. Therefore $fh(D) \cap D = \emptyset$. If $h(z) \in [(\text{Int } r_2) \cup (\text{Int } r_3)]^c$, then $fh(z) \in D_2^c$. Hence $fh(D) \cap D = \emptyset$.

v) In general case, iii) and iv) leads to our conclusion.

Next we shall prove that any point in $R(\Gamma)$ is Γ -equivalent to some point in D . It is clear that $R(G) \supset R(\Gamma)$. It remains to show that a point in $(\text{Int } r_1) \cap (\text{Ext } r_2) \cap (\text{Ext } r_3)$ is Γ -equivalent to some point in D . Let z be in $(\text{Int } r_1) \cap (\text{Ext } r_2) \cap (\text{Ext } r_3)$. When $f(z) \in D$, we have nothing to prove. If $f(z) \notin D$, then $f(z) \in D_2 \setminus D \setminus [(\text{Int } r_2) \cap (\text{Ext } r_1)] \setminus [(\text{Int } r_3) \cap (\text{Ext } r_1)]$. There exist an element h in G and a point x in D_1 such that $h(x) = f(z)$. It is easy to show that $gf(z) \in (\text{Int } r_3)$. Since $(\text{Int } r_3)$ is not contained in D_1 , there are an element k and a point y in D_1 such that $k(y) = gf(z)$. It follows that $gf(z) = gh(x) = k(y)$. Now we note that $x \neq y$. Therefore x and y are G -equivalent in D_1 . This is a contradiction. Thus $f(z)$ is contained in D . Q. E. D.

3. Application

Using our theorem, we can construct a fuchsian group of the first kind from a fuchsian group of the second kind.

Let G be a fuchsian group of the second kind with one funnel. Let g be a hyperbolic element in G corresponding to the funnel. Let $f \in G$ be an elliptic element

with the period 2 whose isometric circle is an axis of g . We denote a Ford polygon for G (resp. $\langle f \rangle$) by R_1 (resp. R_2). Then Γ , the group by G and f , is a fuchsian group of the first kind and $R_1 \cap R_2$ is a Ford polygon for Γ .

reference

- [1] L. R. Ford, Automorphic functions, 2nd ed., Chelsea, 1951.

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